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REMARKS ON BOUNDARIES OF CAT(0) SPACES FROM SHAPE THEORY

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1. Introduction and preliminaries

In this paper, we follow notations and terminologies of [2]. A metric space (X, d) is said to be *proper* if all closed, bounded sets in (X, d) are compact. A metric space (X, d) is said to be a *geodesic space* if for any $x, y \in X$, there exists an isometric embedding $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such a ξ is called a *geodesic*). Let (X, d) be a geodesic space and let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with same edge lengths as T. Choose two points x and y in T. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is called the CAT(0)-*inequality*, where $d_{\mathbb{R}^2}$ is the usual metric on $\mathbb{R}^2$. A geodesic space X is called a CAT(0) *space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T. See for details of CAT(0) spaces in [2, p.158].

Let (X, d) be a proper CAT(0) space. Fix $x_0 \in X$. Set $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$ and $S(x_0, r) = \{x \in X : d(x_0, x) = r\}$. Denote the geodesic segment from x and x' in X by $[x, x']$. There exists the projection $p_r : X \to \overline{B}(x_0, r)$ such that $p_r|_{\overline{B}(x_0,r)} = id$ and $p_r(x) = x'$ if $x \notin \overline{B}(x_0, r)$, where $\{x'\} = S(x_0, r) \cap [x_0, x]$. Let $\overline{X} = \varprojlim(\overline{B}(x_0, n), p_n|_{\overline{B}(x_0,n+1)})$ and $\partial X = \varprojlim(S(x_0, n), r_n)$, said to be *the boundary of* X where $r_n = p_n|_{S(x_0,n+1)} : S(x_0, n + 1) \to S(x_0, n)$ for each $n \in \mathbb{N}$. It is clear that $\overline{X} = X \cup \partial X$ is a compactification of X with a reminder ∂X which is AR (see [12, Lemma 1.1]). It is known that the boundary ∂X of X is independent on the choice of $x_0 \in X$. See for details in [2, pp.263-265].

Definition 1.1 ([4]). Let X and Y be ANR proper metric spaces. A homotopy equivalence $f : X \to Y$ is said to be a *simple homotopy equivalence* if there exist an ANR proper metric space Z and proper cell-like maps $\alpha : Z \to X$, $\alpha' : Z \to Y$ such that $f \circ \alpha$ is proper homotopic to α', written $f \circ \alpha \simeq_p \alpha'$.

Let (X_i, d_i) be a proper CAT(0) space for $i = 0, 1$. First, we show that there exists a simple homotopy equivalence from X_0 to X_1 if and only if ∂X_0 and ∂X_1 are shape equivalent (see Proposition 2.3 below).

Definition 1.2. An *action* of a group Γ on a space X, written $\Gamma \curvearrowright X$, is a homomorphism from Γ to the group of self-homeomorphism of X.

A group Γ is said to *act geometrically on* a metric space (X, d), written $\Gamma \underset{geo.}{\curvearrowright} X$, if $\Gamma \curvearrowright X$ satisfies the following:

(1) (isometry) We have $d(x, x') = d(\gamma x, \gamma x')$ for any $x, x' \in X$ and each $\gamma \in \Gamma$, written $\Gamma \underset{iso.}{\curvearrowright} X$;
(2) (cocompact) There exists a compact subset C of X such that $X = \bigcup_{\gamma \in \Gamma} \gamma C$, written $\Gamma \underset{coc.}{\curvearrowright} X$;
(3) (proper) For every $x \in X$ there exists $\epsilon > 0$ such that $\{\gamma \in \Gamma : \overline{B}(x, \epsilon) \cap \gamma \overline{B}(x, \epsilon) \neq \emptyset\}$ is finite, written $\Gamma \underset{pro.}{\curvearrowright} X$.

Let Γ be a group and let X and Y be spaces with $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$. A map $f : X \to Y$ is said to be Γ-*map* if $f(\gamma x) = \gamma f(x)$ for each $x \in X$ and each $\gamma \in \Gamma$. Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ is said to be Γ-*homotopic* if there exists a Γ-map $H : X \times [0, 1] \to Y$ which is a homotopy from f_0 to f_1.

Gromov [10, Chapter 6] asks whether the visual boundary ∂X_0 of X_0 is Γ-equivariantly homeomorphic to the visual boundary ∂X_1 of X_1 whenever a group Γ acts geometrically on a CAT(0) space X_i. Recall that Γ acts on ∂X_i (see Remark 2.1 below). But, in general, C. B. Croke and B. Kleiner [7] showed that ∂X_0 is not homeomorphic to ∂X_1. By use of a polyhedral resolution of boundaries, P. Ontaneda [12] proved that there exists a proper Γ-homotopy equivalence map $f : X_0 \to X_1$ and ∂X_0 and ∂X_1 are shape equivalent. Then, the map f induces a shape isomorphism $\mathbf{f}$ from ∂X_0 to ∂X_1 and every $\gamma \in \Gamma$ induces a shape isomorphism γ_{X_i} from ∂X_i to ∂X_i (see Remark 2.2 below). In particular, Bestvina posed the following: Are ∂X_0 and ∂X_1 cell-like equivalent? Recall that ∂X_0 and ∂X_1 is said to be *cell-like equivalent* if there exist a compact metric space Z and two cell-like maps $f_i : Z \to \partial X_i$ $(i = 0, 1)$. It is clear that if two compact ANR metric spaces are simple homotopy equivalent, they are cell-like equivalent. By Proposition 2.3 below, we see that $f : X_0 \to X_1$ is a simple homotopy equivalence. In this paper, we state the following result.

Proposition 1.3. *Let Γ be a group and for $i = 0, 1$ let (X_i, d_i) be a proper* CAT(0) *space with $\Gamma \underset{geo.}{\curvearrowright} X_i$. Then there exists a Γ-homotopy equivalence $f : X_0 \to X_1$ with a proper Γ-homotopy inverse $g : X_1 \to X_0$ such that f is a simple homotopy equivalence, $f|_{X_0^G} : X_0^G \to X_1^G$ is a proper homotopy equivalence with a proper homotopy inverse $g|_{X_1^G} : X_1^G \to X_0^G$ for each subgroup G of Γ, and, $\mathbf{f}\gamma_{X_0} = \gamma_{X_1}\mathbf{f}$ for each $\gamma \in \Gamma$.*

Here, $X^G = \{x \in X : \gamma x = x \text{ for all } \gamma \in G\}$ for a subgroup G of Γ.

2. Shape equivalences

Remark 2.1. Let (X, d) be a proper CAT(0) space. Let Γ be a group with $\Gamma \underset{iso.}{\curvearrowright} X$. Since $\gamma : X \to X : x \mapsto \gamma x$ is an isometry for each $\gamma \in \Gamma$, there exists the extension $\overline{\gamma} : \overline{X} \to \overline{X}$ of γ which is a homeomorphism (see [2, Corollary 8.9]). Thus, we have a homeomorphism $\gamma = \overline{\gamma}|_{\partial X} : \partial X \to \partial X$ for each $\gamma \in \Gamma$. Fix $x_0 \in X$. The map γ induces a shape morphism $\gamma_X = (\gamma_{X,n}, \phi) : (S(x_0, n), r_n) \to (S(x_0, n), r_n)$ such that $\overline{\gamma}(X_{\phi(n)}) \subset X_n$ for each $n \in \mathbb{N}$ and $\gamma(\overline{x}) = \lim_{n\to\infty} \gamma_{X,n}(\overline{p}_{\phi(n)}(\overline{x}))$ for each $\overline{x} \in \partial X$, where $X_n = \{x \in X : d(x_0, x) \geq n\}$, $\gamma_{X,n} = p_n \circ \overline{\gamma}|_{S(x_0,\phi(n))} : S(x_0, \phi(n)) \to S(x_0, n)$ and $\overline{p}_n : \overline{X} \to \overline{B}(x_0, n)$ is the extension of p_n for each $n \in \mathbb{N}$. See [11].

Remark 2.2. Let (X_i, d_i) be a proper CAT(0) space. Fix $x_i \in X_i$ for $i = 0, 1$. By Remark 2.1, we have $\partial X_i = \varprojlim(S(x_i, n), r_{i,n})$, where $r_{i,n} = p_{i,n}|_{S(x_i,n+1)} : S(x_i, n + 1) \to S(x_i, n)$ for each $n \in \mathbb{N}$. By [1], we have that ∂X_0 and ∂X_1 are shape equivalent if and only if there exist two functions $\psi, \psi' : \mathbb{N} \to \mathbb{N}$, maps $f_n : S(x_0, \psi^n(1)) \to S(x_0, \psi'^n(1))$, and, $g_n : S(x_0, \psi'^{n+1}(1)) \to S(x_0, \psi^n(1))$ satisfying the following homotopy commutative diagram:

$$\begin{tikzcd}
S(x_0, \psi(1)) \arrow[d, "f_0"'] & S(x_0, \psi^2(1)) \arrow[l, "\pi_1"'] \arrow[d, "f_1"] & S(x_0, \psi^3(1)) \arrow[l, "\pi_2"'] \arrow[d, "f_2"] & \cdots \arrow[l, "\pi_3"'] \\
S(x_1, \psi'(1)) & S(x_1, \psi'^2(1)) \arrow[l, "\pi_1'"] \arrow[ul, "g_1"] & S(x_1, \psi'^3(1)) \arrow[l, "\pi_2'"] \arrow[ul, "g_2"] & \cdots, \arrow[l, "\pi_3'"] \arrow[ul, "g_3"]
\end{tikzcd}$$

where $\pi_k = r_{0,\psi^k(1)} \circ \cdots \circ r_{0,\psi^{k+1}(1)-1}$ and $\pi'_k = r_{1,\psi'^k(1)} \circ \cdots \circ r_{1,\psi'^{k+1}(1)-1}$.

Let $f : X_0 \to X_1$ be a proper homotopy equivalence with a proper homotopy inverse $g : X_1 \to X_0$. Then it is easy to construct shape morphisms $\mathbf{f} = (f_n, \psi) : (S(x_0, \psi^n(1)), r_{0,n}, \mathbb{N}) \to (S(x_0, \psi'^n(1)), r_{1,n}, \mathbb{N})$ and $\mathbf{g} = (g_n, \psi') : (S(x_0, \psi'^n(1)), r_{1,n}, \mathbb{N}) \to (S(x_0, \psi^n(1)), r_{0,n}, \mathbb{N})$, induced by f and g, respectively which satisfy the above. In particular, if $f : X_0 \to X_1$ is a proper Γ-map, $\mathbf{f}\gamma_{X_0} = \gamma_{X_1}\mathbf{f}$ for each $\gamma \in \Gamma$.

Let Q be the Hilbert cube, i.e., $[-1, 1]^\infty$.

Proposition 2.3. *Let (X_i, d_i) be a proper* CAT(0) *space for $i = 0, 1$. The following are equivalent:*

(1) *There exists a proper homotopy equivalence map $f : X_0 \to X_1$;*
(2) *∂X_0 and ∂X_1 are shape equivalent;*
(3) *$X_0 \times Q$ and $X_1 \times Q$ are homeomorphic;*
(4) *There exists a simple homotopy equivalence map $f' : X_0 \to X_1$.*

In particular, every proper homotopy equivalence map from X_0 to X_1 is a simple homotopy equivalence.

Proof. Let $incl_i : X_i = X_i \times \{0\} \hookrightarrow X_i \times Q$ be the inclusion and let $\alpha_i : X_i \times Q \to X_i$ be the projection.

(1) $\Longrightarrow$(2): See Remark 2.2.

(3) $\Longrightarrow$(1): Let $h : X_0 \times Q \to X_1 \times Q$ be a homeomorphism. Thus, we have two proper maps $f = \alpha_2 \circ h \circ incl_1 : X_0 \to X_1$ and $g = \alpha_1 \circ h^{-1} \circ incl_2 : X_1 \to X_0$ such that $g \circ f$ is proper homotopic to the identity map id_{X_0} and $f \circ g$ is proper homotopic to the identity map id_{X_1}.

(2) $\Longrightarrow$(3): Let $\overline{X_i} = X \cup \partial X_i$ which is AR for $i = 0, 1$. By [4], $\overline{X_i} \times Q$ is homeomorphic to Q. Since $\partial X_i \times Q$ is a Z-set in $\overline{X_i} \times Q$ for $i = 0, 1$, by [4, Theorem 25.2], $X_0 \times Q$ is homeomorphic to $X_1 \times Q$.

(1) $\Longleftrightarrow$(4): It suffices to show (1) $\Longrightarrow$(4). Let f be a proper homotopy equivalence. By [6, Theorem 7], there exists a homeomorphism $h : X_0 \times Q \to X_1 \times Q$ which is proper homotopic to $f \times \mathrm{id}_Q : X_0 \times Q \to X_1 \times Q$. Let $\alpha_i : X_i \times Q \to X_i$ be the projection for $i = 0, 1$. By a proper homotopy commutative diagram

$$\begin{array}{ccc}
X_0 \times Q & \xrightarrow{h} & X_1 \times Q \\
{\scriptstyle \mathrm{id}_{X_0 \times Q}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X_1 \times Q}} \\
X_0 \times Q & \xrightarrow{f \times \mathrm{id}_Q} & X_1 \times Q \\
{\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\
X_0 & \xrightarrow{f} & X_0
\end{array}$$

we have $f \circ \alpha_0 \simeq_p \alpha_1 \circ h$, thus f is a simple homotopy equivalence. □

Example 2.4. For $i = 0, 1$ let Z_i be a continuum such that Z_0 and Z_1 are shape equivalent. By [3] or [9], for $i = 0, 1$ there exists a proper CAT(0) space (X_i, d_i) such that ∂X_i is homeomorphic to Z_i. By Proposition 2.3, X_0 and X_1 are simple homotopy equivalent.

3. The existence of proper map

Let Γ be a group and for $i = 0, 1$ let (X_i, d_i) be a proper CAT(0) space with $\Gamma \underset{geo.}{\curvearrowright} X_i$. In [12, Theorem C], it was proved that there exists a proper Γ-homotopy equivalence $f : X_0 \to X_1$. But, in this section we give a more direct proof by no use of of a polyhedral resolution of boundaries.

Lemma 3.1. *Let Γ be a group, let (X, d) be a proper CAT(0) space with $\Gamma \underset{geo.}{\curvearrowright} X$, and, let $f : X \to X$ be a proper Γ-map. Then there exists a proper Γ-homotopy $H : X \times [0, 1] \to X$ from f to the identity map id_X. In particular, for every subgroup G of Γ, $H|_{X^G} : X^G \times [0, 1] \to X^G$ is a proper homotopy from $f|_{X^G} : X^G \to X^G$ to the identity map id_{X^G}.*

Sketch of proof. Since $\Gamma \underset{geo.}{\curvearrowright} X$ and f is a Γ-map, there exists $r > 0$ such that $d(f, \mathrm{id}_X) < r$. For every $x \in X$ Let $c_x : [0, d(f(x), x)] \to X$ be a geodesic connecting from $f(x)$ to x. Define $H : X \times [0,1] \to X$ by $H(x,t) = c_x(td(f(x),x))$ for each $x \in X$ and each $t \in [0,1]$. It is clear that H is a proper homotopy from f to id_X. In particular, if $f : X \to X$ is a Γ-map, so is H. □

Definition 3.2. [2, p. 179] Let (X, d) be a metric space, let Y be a bounded set of X and let Z be a closed subset of X. The *radius* of Y at Z, is defined by

$$r_Z(Y) = \inf\{r > 0 : x \in Z, Y \subset \overline{B}(x,r)\}.$$

For simplicity of notation, if $X = Z$, we write $r(Y)$ instead of $r_X(Y)$.

Proposition 3.3. [2, Proposition II 2.7] *Let (X, d) be a complete* CAT(0) *space, let Y be a bounded set of X and let Z be a closed convex subset of X. Then there exists a unique point $c_Z(Y) \in Z$, called the centre of Y at Z, such that $Y \subset \overline{B}(c_Z(Y), r_Z(Y))$.*

Sketch of proof. There exist a sequence $\{z_n\}_{n\in\mathbb{N}}$ of Z and $\{r_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}_+$ such that $r_Z(Y) = \lim_{n\to\infty} r_n$ and $Y \subset \overline{B}(z_n, r_n)$ for all $n \in \mathbb{N}$. We can show that for every $\epsilon > 0$ there exist $R, R' > 0$ with $R > r_Z(Y) > R' > 0$ such that $\operatorname{diam}[z_n, z_{n'}] < 2\epsilon$ for any $n, n' \in \mathbb{N}$ with $r_n, r_{n'} < R$. This shows that $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, so $c_Z(Y) = \lim_{n\to\infty} z_n$, and establishes the uniqueness of $c_Z(Y)$. □

Lemma 3.4. *Let Γ be a group and let (X, d) be a complete* CAT(0) *space with $\Gamma \underset{iso.}{\curvearrowright} X$. Then $X^G = \{x \in X : \gamma x = x \text{ for all } \gamma \in G\}$ is a convex set for each subgroup G of Γ. In particular, X^G is a nonempty convex set for each finite subgroup G of Γ.*

Sketch of proof. Fix $x, x' \in X^G$. Let $\xi : [0, d(x,x')] \to X$ be a geodesic from x to x'. Since $\xi(2^{-1}d(x,x')) \in X^G$, we have $\{\xi(2^{-n}kd(x,x')) : n, k \in \mathbb{N}, 0 \le k \le 2^n\} \subset X^G$, thus, $\xi([0, d(x,x')]) \subset X^G$. Let G be a finite subgroup of Γ and fix $x_0 \in X$. By Proposition 3.4, $c(Gx_0) \in X^G$, thus it is nonempty. □

Definition 3.5. Let Γ be a group and let $K = |\mathcal{K}|$ be a simplicial complex with $\Gamma \curvearrowright K$. Set $\Gamma^x = \{\gamma \in \Gamma : \gamma x = x\}$ for $x \in K$ and $\Gamma^A = \bigcap_{y\in A} \Gamma^y$ for $A \subset K$. $\Gamma \curvearrowright K$ is *simplicial* if it is satisfied the following;

(1) $\gamma : K \to K$ is a simplicial map for each $\gamma \in \Gamma$;
(2) $\Gamma^\sigma = \{\gamma \in \Gamma : \gamma\sigma = \sigma\}$ for each $\sigma \in \mathcal{K}$.

The proof of the following result is based on the proof of [8, p.286, Theorem A.2].

Lemma 3.6. *Let Γ be a group, let (X,d) be a proper* CAT(0) *space with $\Gamma \underset{geo.}{\curvearrowright} X$, and, let K be a locally finite simplicial complex with $\Gamma \underset{coc.,pro.}{\curvearrowright} K$ such that $\Gamma \curvearrowright X$ is simplicial. Then, for every Γ-invariant subcomplex L of K and every proper Γ-map $f : L \to X$, there exists a proper Γ-map $\widetilde{f} : K \to X$ such that $\widetilde{f}|_L = f$.*

Proof. Let $\mathcal{K}$ be a subdivision of K and let $\mathcal{K}^{(n)}$ be the n-skeleton of $\mathcal{K}$. We show by induction on n that for every proper Γ-map $f_n : L \cup |\mathcal{K}^{(n)}| \to X$, there exists a proper Γ-map $f_{n+1} : L \cup |\mathcal{K}^{(n+1)}| \to X$ such that $f_{n+1}|_{L\cup|\mathcal{K}^{(n)}|} = f_n$.

By assumption, there exists a finite subset S_0 of $|\mathcal{K}^{(0)}| \setminus L$ such that $\Gamma S_0 = |\mathcal{K}^{(0)}| \setminus L$, and, $\Gamma v \cap S_0 = \{v\}$ for each $v \in S_0$. Since $\Gamma \underset{pro.}{\curvearrowright} K$, $\Gamma^v = \{\gamma \in \Gamma : \gamma v = v\}$ is a finite subgroup of Γ for each $v \in S_0$. By Lemma 3.5, $X^{\Gamma^v} = \{x \in X : \gamma x = x \text{ for all } \gamma \in \Gamma^v\}$ is nonempty for each $v \in S_0$. Choose $\widetilde{v} \in X^{\Gamma^v}$. Let us define $f_0 : L \cup |\mathcal{K}^{(0)}| \to X$ by $f_0|_L = f$ and $f(\gamma v) = \gamma\widetilde{v}$ for each each $v \in S_0$ and each $\gamma \in \Gamma$. Let $\gamma, \gamma' \in \Gamma$ and $v, v' \in S_0$ with $\gamma v = \gamma' v'$. We show that $\gamma\widetilde{v} = \gamma'\widetilde{v'}$. Since $\Gamma v \cap S_0 = \{v\}$ for each $v \in S_0$, we have $v = v'$, thus, $\gamma^{-1}\gamma' \in \Gamma^v$. Hence, $\gamma^{-1}\gamma'\widetilde{v} = \widetilde{v}$, and finally that $\gamma\widetilde{v} = \gamma'\widetilde{v'}$. Therefore, f_0 is well-defined and a Γ-map. We show that f_0 is a proper map, i.e., $f_0^{-1}(Z)$ is compact for each compact set $Z \subset X$. Let $\Gamma_Z(v) = \{\gamma \in \Gamma : \gamma f_0(v) \in Z\}$ for each $v \in S_0$. Since $\Gamma \underset{pro.}{\curvearrowright} X$, $\Gamma_Z(v)$ is finite. Since $f_0^{-1}(Z) \subset f^{-1}(Z) \cup \bigcup\{\gamma v : v \in S_0, \gamma \in \Gamma_Z(v)\}$, $f_0^{-1}(Z)$ is compact.

Let $f_n : L \cup |\mathcal{K}^{(n)}| \to X$ be a proper Γ-map for $n \geq 0$. By assumption, there exists a finite subset S_{n+1} of $\mathcal{K}^{(n+1)} \setminus \mathcal{K}^{(n)}$ such that $\Gamma(\bigcup_{\sigma\in S_{n+1}} \mathrm{int}\sigma) = |\mathcal{K}^{(n+1)}| \setminus (L \cup |\mathcal{K}^{(n)}|)$, and, $\Gamma(\mathrm{int}\sigma) \cap \bigcup_{\sigma\in S_{n+1}} \sigma = \mathrm{int}\sigma$ for each $\sigma \in S_{n+1}$, where $\partial\sigma = \bigcup\{\tau : \tau \text{ is a proper face of } \sigma\}$ and $\mathrm{int}\sigma = \sigma \setminus \partial\sigma$. Let $\sigma \in S_{n+1}$. Recall $\Gamma^\sigma = \{\gamma \in \Gamma : \gamma z = z \text{ for each } z \in \sigma\}$. Since Γ^z is finite and $\Gamma^\sigma \subset \Gamma^z$ for each $z \in \sigma$, Γ^σ is a finite subgroup of Γ. It is clear that $f(\partial\sigma) \subset X^{\Gamma^\sigma} = \{x \in X : \gamma x = x \text{ for all } \gamma \in \Gamma^\sigma\}$. By Proposition 3.4, we have the centre $c(f(\partial\sigma))$ of $f(\partial\sigma)$ in X. Since $\Gamma \underset{iso.}{\curvearrowright} X$, by Proposition 3.4, we see that $c(f(\partial\sigma)) \in X^{\Gamma^\sigma}$. Set $c(f(\partial\sigma)) * f(\partial\sigma) = \bigcup\{[c(f(\partial\sigma)), x] : x \in f(\partial\sigma)\}$. Let $c(\sigma)$ be the barycenter of σ and let $f_{n+1,\sigma} : \sigma = c(\sigma) * \partial\sigma \to c(f(\partial\sigma)) * f(\partial\sigma) \subset X$ be the cone on $f_n|_{\partial\sigma}$. By Lemma 3.5, X^{Γ^σ} is a convex subset of X, so $f_{n+1,\sigma}(\sigma) \subset X^{\Gamma^\sigma}$. Define a map $f_{n+1} : L \cup |\mathcal{K}^{(n+1)}| \to X$ satisfying $f_{n+1}|_{L\cup|\mathcal{K}^{(n)}|} = f_n$ by $f_{n+1}(\gamma z) = \gamma f_{n+1,\sigma}(z)$ for each $\sigma \in S_{n+1}$, each $z \in \mathrm{int}\sigma$, and, each $\gamma \in \Gamma$. Let $\gamma, \gamma' \in \Gamma$, $\sigma, \sigma' \in S_{n+1}$, and, $z \in \mathrm{int}\sigma, z' \in \mathrm{int}\sigma'$ with $\gamma z = \gamma' z'$. We show that $f_{n+1}(\gamma z) = f_{n+1}(\gamma' z')$. By the definition of S_{n+1}, we see $\sigma = \sigma'$. Since $\Gamma \curvearrowright X$ is simplicial, we have $\gamma^{-1}\gamma' \in \Gamma^\sigma$, hence, $z = z'$. Since $f_{n+1,\sigma}(\sigma) \subset X^{\Gamma^\sigma}$, we have $\gamma^{-1}\gamma' f_{n+1,\sigma}(z) = f_{n+1,\sigma}(z)$, hence, $f_{n+1}(\gamma z) = f_{n+1}(\gamma' z')$. Therefore, f_{n+1} is well-defined and a Γ-map.

We show that f_{n+1} is a proper map, i.e., $f_{n+1}^{-1}(Z)$ is compact for each compact set $Z \subset X$. Let $\Gamma_Z(\sigma) = \{\gamma \in \Gamma : \gamma f_0(\sigma) \in Z\}$ for each $\sigma \in S_{n+1}$. Since $\Gamma \underset{pro.}{\curvearrowright} X$, $\Gamma_Z(\sigma)$ is finite. Since $f_{n+1}^{-1}(Z) \subset f^{-1}(Z) \cup \bigcup\{\gamma v : \sigma \in S_{n+1}, \gamma \in \Gamma_Z(\sigma)\}$, $f_{n+1}^{-1}(Z)$ is compact. □

We show the following lemma, and it directly follows from [12, Proposition A], but we give a more direct proof based on the proof of it.

Lemma 3.7. *Let* Γ *be a group and for* $i = 0, 1$ *let* (X_i, d_i) *be a proper* CAT(0) *space with* $\Gamma \underset{geo.}{\curvearrowright} X_i$. *Then there exists a proper* Γ*-map* $f : X_0 \to X_1$

Proof. By $\Gamma \underset{coc.}{\curvearrowright} X_0$, there exist a compact set C of X_0 such that $\Gamma C = X_0$. By [2, Proposition I.8.5(1)], for every $x \in C$ there exists $\epsilon_x > 0$ such that every $\gamma \in \Gamma$,

$$\gamma x = x \text{ or } \overline{B}(x, \epsilon_x) \cap \gamma \overline{B}(x, \epsilon_x) = \emptyset. \tag{1}$$

Thus, there exist a finite subset $X_0' = \{x_0, \dots, x_l\}$ of C such that $\Gamma \mathcal{V}$ is a locally finite open cover of X_0 and $U \not\subset \bigcup\{U' \in \Gamma\mathcal{V} : U \neq U'\}$ for each $U \in \Gamma\mathcal{V}$, where $\mathcal{V} = \{B(x_i, \epsilon_{x_i}) : i = 0, \dots, l\}$.

Let $\mathcal{L}$ be the nerve of $\Gamma\mathcal{V}$, i.e., $\mathcal{L}^{(0)} = \mathcal{U}$, and, $\langle U_0, \dots, U_k \rangle \in \mathcal{L}$ if and only if $U_0 \cap \dots \cap U_k \neq \emptyset$. Set $L = |\mathcal{L}|$. For every $\gamma \in \Gamma$, define a simplicial map $\gamma : L \to L$ by $\gamma(\langle U_0, \dots, U_k \rangle) = \langle \gamma U_0, \dots, \gamma U_k \rangle$ for each $\langle U_0, \dots, U_k \rangle \in \mathcal{L}$. Since $U = \gamma U$ whenever $U \cap \gamma U \neq \emptyset$, we have $\Gamma \curvearrowright L$.

Let $\gamma \in \Gamma$ and $\langle U_0, \dots, U_k \rangle \in \mathcal{L}$ such that $\gamma(\langle U_0, \dots, U_k \rangle) = \langle U_0, \dots, U_k \rangle$, i.e., $\{U_0, \dots, U_k\} = \{\gamma U_0, \dots, \gamma U_k\}$. Since $\bigcap_{i=0}^k U_i = \bigcap_{i=0}^k \gamma U_i \neq \emptyset$, we have $U_i \cap \gamma U_i \neq \emptyset$, hence, $U_i = \gamma U_i$ for each $i = 0, \dots, k$. Therefore, $\Gamma \curvearrowright L$ is simplicial.

We show that $\Gamma \underset{coc.}{\curvearrowright} L$. Let $\mathcal{T} = \{\langle V_0, \dots, V_k \rangle \in \mathcal{L} : V_i \in \mathcal{V}$ for each $i\}$ such that $|\mathcal{T}|$ is a finite subcomplex of L. It suffices to show that $L = \Gamma|St(\mathcal{T}, \mathcal{L})|$, where $St(\mathcal{T}, \mathcal{L}) = \{\sigma \in \mathcal{L} : \sigma \cap |\mathcal{T}| \neq \emptyset\}$ is the close star of $\mathcal{T}$ in $\mathcal{L}$. Let $\langle \gamma_0 V_0, \dots, \gamma_k V_k \rangle \in \mathcal{L}$ such that $\gamma_i \in \Gamma$ and $V_i \in \mathcal{V}$ for each $i = 0, \dots, k$. Since $\gamma_0 V_0 \cap \dots \cap \gamma_k V_k \neq \emptyset$, we have $V_0 \cap \gamma_0^{-1}\gamma_1 V_1 \cap \dots \cap \gamma_0^{-1}\gamma_k V_k \neq \emptyset$. Since $V_0 \in \mathcal{T}^{(0)}$, we have $\langle V_0, \gamma_0^{-1}\gamma_1 V_1, \dots, \gamma_0^{-1}\gamma_k V_k \rangle \in St(\mathcal{T}, \mathcal{L})$. Since $\langle \gamma_0 V_0, \dots, \gamma_k V_k \rangle = \gamma_0 \langle V_0, \gamma_0^{-1}\gamma_1 V_1, \dots, \gamma_0^{-1}\gamma_k V_k \rangle \in \gamma_0 St(\mathcal{T}, \mathcal{L})$, we have $|\langle \gamma_0 V_0, \dots, \gamma_k V_k \rangle| \in \Gamma|St(\mathcal{L}, \mathcal{K})|$, thus, $L = \Gamma|St(\mathcal{T}, \mathcal{L})|$. By the above, we see that $\dim L = \dim |St(\mathcal{T}, \mathcal{L})| < \infty$.

We show that $\Gamma \underset{pro.}{\curvearrowright} L$. Since $\mathcal{L}^{(0)} = \Gamma\mathcal{T}^{(0)}$, it suffices to show that for any $V \in \mathcal{V}$, $\{\gamma \in \Gamma : |St(V, \mathcal{L})| \cap \gamma|St(V, \mathcal{L})| \neq \emptyset\}$ is finite. This follows that $\{\gamma \in \Gamma : V \cap \gamma V' \neq \emptyset\}$ is finite for each $V' \in \mathcal{V}$ with $\gamma' \in \Gamma$ and $V \cap \gamma' V' \neq \emptyset$.

We construct the canonical map $f_0 : X_0 \to L$. Let $x \in X_0$. Set $\{U \in \Gamma\mathcal{V} : x \in U\} = \{U_0, \dots, U_k\}$. Define

$$\lambda_i(x) = \frac{d(x, X_0 \setminus U_i)}{\sum_{j=0}^k d(x, X_0 \setminus U_j)} \text{ and } f_0(x) = \sum_{i=0}^k \lambda_i(x) U_i \in \langle U_0, \dots, U_k \rangle.$$

Since $f_0^{-1}(\langle U_0, \dots, U_k \rangle) \subset U_0 \cup \dots \cup U_k$, we see that f_0 is a proper map. Since $\gamma : X_0 \to X_0$ is an isometry, for every $\gamma \in \Gamma$ we have

$$\lambda_i(\gamma x) = \frac{d(\gamma x, X_0 \setminus \gamma U_i)}{\sum_{j=0}^k d(\gamma x, X_0 \setminus \gamma U_j)} = \frac{d(x, X_0 \setminus U_i)}{\sum_{j=0}^k d(x, X_0 \setminus U_j)} = \lambda_i(x),$$

thus, since $\gamma : L \to L$ is a simplicial map,

$$f_0(\gamma x) = \sum_{i=0}^{k} \lambda_i(\gamma x)\gamma U_i = \sum_{i=0}^{k} \lambda_i(x)\gamma U_i = \gamma\left(\sum_{i=0}^{k} \lambda_i(x)U_i\right) = \gamma f_0(x),$$

thus, f_0 is a Γ-map.

By Lemma 3.7, there exists a proper Γ-map $f_1 : L \to X_1$, therefore, we have a a proper Γ-map $f = f_1 \circ f_0 : X_0 \to X_1$, which completes the proof. □

Let L be as in the proof of Lemma 3.8. We can think of L as a piecewise Euclidean complex, a locally finite simplicial complex with the intrinsic pseudometric ρ (see [2, pp.98-99]) such that a length of every 1-simplex in $\mathcal{L}$ is one. Since Shape(L) is finite (see [2, p.98]), (L, ρ) is a complete geodesic space ([2, Theorem I .7.19, p.105]). In particular, by the construction of (L, ρ), $\gamma : (L, \rho) \to (L, \rho)$ is an isometry for each $\gamma \in \Gamma$, i.e., $\Gamma \underset{iso.}{\curvearrowright} L$.

The proof of Proposition 1.3. By Lemma 3.8, for $i = 0, 1$ there exist proper Γ-maps $f : X_0 \to X_1$ and $g : X_1 \to X_0$. By Remark 2.2, Proposition 2.3 and Lemma 3.1, f and g satisfy the conditions in Proposition 1.3, which completes the proof. □

4. Questions

Question 4.1. *Let Γ be a group, let (X_i, d) be a proper* CAT(0) *space with $\Gamma \underset{geo.}{\curvearrowright} X_i$, and let $f : X_0 \to X_1$ be a proper Γ-map. Does there exist an ANR proper metric space Z with $\Gamma \underset{geo.}{\curvearrowright} Z$ and proper cell-like Γ-maps $\alpha : Z \to X_0$, $\alpha' : Z \to X_1$ such that $f \circ \alpha$ is proper Γ-homotopic to α'?, i.e., is $f : X_0 \to X_1$ a simple Γ-homotopy equivalence?*

Question 4.2. *Let Γ be a group and let (X, d) be a proper* CAT(0) *space with $\Gamma \underset{geo.}{\curvearrowright} X$. If there exists a compact ANR metric space Z which is shape (Γ-)equivalent to ∂X, is ∂X ANR?*

Question 4.3. *Let Γ be a group and let (X_i, d) be a proper* CAT(0) *space with $\Gamma \underset{geo.}{\curvearrowright} X_i$ such that ∂X_i is ANR for $i = 0, 1$.*

(1) *Does there exists a Γ-homotopy equivalence map from ∂X_0 and ∂X_1?*
(2) *Are ∂X_0 and ∂X_1 simple homotopy equivalent?*

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